

Geometrical Properties of Sections of Buchsbaum-Rim Sheaves

or

How to Construct Gorenstein Schemes of Higher Codimension with *Singular*

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1. Introduction

In this *survey article* we want to discuss one of the ways to construct arithmetically Gorenstein subvarieties of projective space. It is well-known that in codimension 2 an arithmetically Gorenstein subvariety is always a complete intersection. For codimension ≥ 4 , however, the construction of Gorenstein subschemes, apart from complete intersections, is quite a complicated problem since no structure theorem is known as in codimension 3 ([2]). On the other hand, it is necessary to develop a technique for constructing such schemes in view of Gorenstein liaison.

Consider the reflexive kernel sheaves \mathcal{B}_ϕ of sufficiently general, generically surjective morphisms ϕ between decomposable bundles over \mathbb{P}^n , so called Buchsbaum-Rim sheaves. The desired Gorenstein schemes appear quite unexpectedly as the top-dimensional part of the zero-locus of a regular section $s \in H^0(\mathbb{P}^n, \mathcal{B}_\phi)$ ([5], [6]). As an application one gets information about the geometrical properties of sections of certain non-split rank n vector bundles on \mathbb{P}^n .

We show a way how to implement the construction method in the computer algebra system *Singular* and produce some examples of Gorenstein curves and threefolds in \mathbb{P}^6 .

Finally, we investigate a class of sheaves \mathcal{B}_ϕ where the degeneracy locus of ϕ does not have the expected codimension.

2. Gorenstein points in \mathbb{P}^3

Let $R = k[z_0, z_1, z_2, z_3]$, $k = \bar{k}$ and A a homogeneous $t \times (t+3)$ -matrix over R such that ideal of all $t \times t$ -minors of A is (z_0, z_1, z_2, z_3) -primary. A is given as

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a graded degree zero homomorphism between two free R -modules F and G of rank $t + 3$ and t respectively:

$$0 \longrightarrow Q \longrightarrow F \xrightarrow{A} G \longrightarrow \operatorname{coker}(A) \longrightarrow 0.$$

Because of our conditions on the minors, $\operatorname{coker}(A)$ has finite length. Sheafifying the sequence, one gets:

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then \mathcal{Q} is a vector bundle of rank 3. We can assume that \mathcal{Q} has global sections - if necessary twist it. Choose a regular global section s , i.e. one with a zero-dimensional degeneracy locus.

Theorem 1 ([5]) *Let $X = Z(s)$ be the zero locus of s . Then X is arithmetically Gorenstein and its saturated ideal I_X has a free resolution*

$$0 \longrightarrow R(-c_1) \longrightarrow F(-c_1) \oplus G^* \longrightarrow G(-c_1) \oplus F^* \longrightarrow I_X \longrightarrow 0,$$

where $c_1 = c_1(\mathcal{Q})$ denotes the first Chern class of \mathcal{Q} .

Idea of the proof: The section s determines an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{Q} \longrightarrow \mathcal{C} \longrightarrow 0.$$

One can easily show that the cokernel \mathcal{C} is reflexive of rank 2. Hence $\mathcal{C}^* = \mathcal{C}(-c_1)$ where $c_1(\mathcal{Q}) = c_1(\mathcal{C})$ is the first Chern class. Dualizing the above sequence, we get

$$0 \longrightarrow \mathcal{C}(-c_1) \longrightarrow \mathcal{Q}^* \xrightarrow{s^t} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

since $\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_X$ and \mathcal{Q} is locally free. Using $H_*^1(\mathbb{P}^3, \mathcal{C}) = H_*^1(\mathbb{P}^3, \mathcal{Q})$ one obtains a diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & H_*^0(\mathcal{C}(-c_1)) & \longrightarrow & H_*^0(\mathcal{Q}^*) & \longrightarrow & I_X & \longrightarrow H_*^1(\mathcal{C}(-c_1)) \longrightarrow 0 \\
& \uparrow & & \uparrow & & & \uparrow \\
& E_1(-c_1) & & F^* & & & G(-c_1) \\
& \uparrow & & \uparrow & & & \uparrow \\
& R(-c_1) \oplus E_2(-c_1) & & G^* & & & F(-c_1) \\
& \uparrow & & \uparrow & & & \uparrow \\
& E_3(-c_1) & & 0 & & & E_1(-c_1) \\
& \uparrow & & & & & \uparrow \\
& 0 & & & & & E_2(-c_1) \\
& & & & & & \uparrow \\
& & & & & & E_3(-c_1) \\
& & & & & & \uparrow \\
& & & & & & 0
\end{array}$$

where $E_i = \wedge^{t+i} F \otimes S^{i-1}(G)^* \otimes \wedge^t G^*$ are the R -modules from the acyclic (!) Buchsbaum-Rim complex associated to $F \xrightarrow{A} G$. Now the application of a mapping cone, the horseshoe lemma and $\text{hd}_R I_X = 2$ (X is arithmetically Cohen-Macaulay) yield an exact sequence

$$0 \longrightarrow R(-c_1) \oplus E_1(-c_1) \longrightarrow F(-c_1) \oplus G^* \oplus E_1(-c_1) \longrightarrow G(-c_1) \oplus F^* \longrightarrow \dots \longrightarrow I_X \longrightarrow 0$$

Using this sequence, one proves immediately that the h -vector of R/I_X is symmetric. Furthermore, one shows that X has the generalized Cayley-Bacharach property with respect to $|K_{\mathbb{P}_3} \otimes \deg Q|$. But this implies that X is arithmetically Gorenstein and therefore we can delete $E_1(-c_1)$ in the above resolution.

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Corollary 1 ([5]) *Let s be a regular global section of a rank three vector bundle \mathcal{E} on \mathbb{P}^3 with $H_*^2(\mathbb{P}^3, \mathcal{E}) = 0$ but non-vanishing first cohomology. Then the zero scheme $X = Z(s)$ is arithmetically Gorenstein.*

Proof: The module $H_*^1(\mathbb{P}^3, \mathcal{E})$ has finite length by the Enriques-Zariski-Severi vanishing lemma and thus the Auslander-Buchsbaum theorem implies the existence of a minimal free resolution

$$0 \longrightarrow F_4 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F \longrightarrow G \longrightarrow H_*^1(\mathbb{P}^3, \mathcal{E}) \longrightarrow 0.$$

After sheafifying, we get

$$0 \longrightarrow \mathcal{F}_4 \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow 0.$$

From the sequence

$$0 \longrightarrow \ker(\phi) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

and the associated long exact sequence in cohomology we deduce that $\ker(\phi)$ is a vector bundle with $H_*^1(\ker(\phi)) \cong H_*^1(\mathcal{E})$, $H_*^2(\ker(\phi)) = 0$. Thus, $\mathcal{E} = \ker(\phi) \oplus \mathcal{L}$, where \mathcal{L} is a decomposable bundle. But \mathcal{E} does not split by Horrocks' theorem and therefore $\mathcal{E} = \ker(\phi)$. Now apply Theorem 1.

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Example 1: Consider the 2×5 -matrix

$$\phi = \begin{pmatrix} z_0^3 + z_1^3 & z_0^3 + z_2^3 & z_0^3 + z_3^3 & z_1^3 + z_2^3 & z_1^3 + z_3^3 \\ z_2^3 + z_3^3 & z_0^3 & z_1^3 & z_2^3 & z_3^3 \end{pmatrix}$$

as map in the exact sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow 5 \mathcal{O}_{\mathbb{P}^3}(6) \xrightarrow{\phi} 2 \mathcal{O}_{\mathbb{P}^3}(9) \longrightarrow 0.$$

Here $c_1(\mathcal{Q}) = 12$. The ideal I of the vanishing locus of a randomly chosen regular section $s \in H^0(\mathbb{P}^3, \mathcal{Q})$ is for example

$$\begin{aligned} J = & (103 z_1^3 z_2^3 + 66 z_2^6 - 763 z_0^3 z_3^3 + 660 z_1^3 z_3^3 + 829 z_2^3 z_3^3 - 713 z_3^6, \\ & 618 z_0^3 z_2^3 + 90 z_2^6 - 5741 z_0^3 z_3^3 + 5123 z_1^3 z_3^3 + 7067 z_2^3 z_3^3 - 5935 z_3^6, \\ & 103 z_1^6 - 168 z_2^6 + 959 z_0^3 z_3^3 - 959 z_1^3 z_3^3 - 1230 z_2^3 z_3^3 + 1019 z_3^6, \\ & 103 z_0^3 z_1^3 - 174 z_2^6 + 710 z_0^3 z_3^3 - 710 z_1^3 z_3^3 - 987 z_2^3 z_3^3 + 934 z_3^6, \\ & 4326 z_0^6 - 8814 z_2^6 + 12847 z_0^3 z_3^3 - 9139 z_1^3 z_3^3 - 13009 z_2^3 z_3^3 + 18923 z_3^6). \end{aligned}$$

The non-saturated zero-scheme $Z(I)$ of degree 54 has h -vector

$$h_I = (1, 3, 6, 10, 15, 21, 23, 21, 15, 7, -3, -15, -20, -18, -9, -3)$$

and is consequently not Gorenstein. Computing the saturation J of I , we get the Gorenstein ideal

$$(13 z_1^3 + 18 z_2^3 + 31 z_3^3, 546 z_0^3 + 534 z_2^3 + 2471 z_3^3, 5976 z_2^6 - 32430 z_2^3 z_3^3 - 90965 z_3^6)$$

with $\dim Z(J) = 0$ and $\deg Z(J) = 54$ – as expected in this case (cf. Remark 2). The symmetric h -vector is $h_J = (1, 3, 6, 8, 9, 9, 8, 6, 3, 1)$ and a minimal free resolution is:

$$0 \longrightarrow R(-12) \longrightarrow R(-6) \oplus 2 R(-9) \longrightarrow 2 R(-3) \oplus R(-6) \longrightarrow I \longrightarrow 0.$$

Thus, we observe that the free resolution in Theorem 1 is not necessarily minimal.

Remark 1: We want to mention that not *all* Gorenstein zero schemes in \mathbb{P}^3 can be obtained in this way. Indeed, the Buchsbaum-Eisenbud structure theorem [2] tells us that every codimension 3 arithmetically Gorenstein subscheme in \mathbb{P}^n is precisely the zero set of the ideal I generated by the $2N \times 2N$ Pfaffians (= roots of the $2N \times 2N$ principal minors) of some $(2N + 1) \times (2N + 1)$ skew-symmetric matrix over $R = k[z_0, z_1, \dots, z_n]$.

Let $R = k[z_0, z_1, z_2, z_3]$ and

$$A = \begin{pmatrix} 0 & -z_2 & 0 & z_3 & 0 \\ z_2 & 0 & -z_1 & 0 & -z_3 \\ 0 & z_1 & 0 & 0 & -z_2 \\ -z_3 & 0 & 0 & 0 & -z_1 \\ 0 & z_3 & z_2 & z_1 & 0 \end{pmatrix}.$$

A *minimal* generating system of the ideal I is in this case for example

$$(z_1^2, z_2^2, z_1 z_2 - z_3^2, z_1 z_3, z_2 z_3).$$

It defines a non-reduced Gorenstein point X of degree 5 and cannot be obtained via a regular section of a rank 3 Buchsbaum-Rim sheaf. The main reason is that

X cannot be generated scheme-theoretically from 4 equations and is therefore not obtained from a homomorphism

$$\phi : \bigoplus_{i=1}^4 R(a_i) \longrightarrow R(b).$$

The $t \times (t+3)$ matrices with $t > 1$ are also out of the question for degree reasons as we will see below.

Remark 2: Suppose that $\mathcal{F} = \bigoplus_{i=1}^{t+3} \mathcal{O}_{\mathbb{P}_3}(a_i)$, $\mathcal{G} = \bigoplus_{j=1}^t \mathcal{O}_{\mathbb{P}_3}(b_j)$ and let ω denote the hyperplane class. We want to compute the degree of X in terms of the integers a_i and b_j . Clearly, the fundamental class $[X]$ equals $c_3(\mathcal{Q})$. The Chern polynomial of the bundle \mathcal{Q} is

$$c_\omega(\mathcal{Q}) = c_\omega(\mathcal{F})c_\omega(\mathcal{G})^{-1} = \frac{\prod_{i=1}^{t+3}(1 + a_i \omega)}{\prod_{j=1}^t(1 + b_j \omega)} = \sum_{i=0}^3 s_i(a) \omega^i \left(\sum_{j=0}^{\min(t,3)} s_j(b) \omega^j \right)^{-1},$$

where $s_i(a) = s_i(a_1, \dots, a_{t+3})$ and $s_j(b) = s_j(b_1, \dots, b_t)$ respectively denote the elementary symmetric functions. A straightforward computation gives:

$$\deg(X) = \begin{cases} s_3(a) - b_1 s_2(a) + b_1^2 s_1(a) - b_1^3, & t = 1 \\ s_3(a) - (b_1 + b_2) s_2(a) + (b_1^2 + b_1 b_2 + b_2^2) s_1(a) - (b_1^3 + b_1^2 b_2 + b_1 b_2^2 + b_2^3), & t = 2 \\ s_3(a) - s_3(b) - s_1(a) s_2(b) - s_2(a) s_1(b) + 2s_1(b) s_2(b) - s_1(b)^2 s_1(a) - s_1(b)^3, & t \geq 3 \end{cases}$$

Therefore one can use the described method to construct Gorenstein codimension three schemes of a given degree.

3. The General Case

Migliore, Nagel and Peterson extended the above result in [6] to higher, odd codimension. This provides a construction technique for Gorenstein subschemes of \mathbb{P}^n with prescribed degree in cases where no structure theorem is known, as for example in codim 5, 7, 9, ...

One can also arrange that the new subschemes contain a given equidimensional subscheme of the same codimension. This is very useful from the viewpoint of Gorenstein liaison.

The setup in the general situation is the following:

Let $Z = \text{Proj}(R)$, where R is a graded Gorenstein k -algebra of $\dim(R) = n + 1$. Let $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of vector bundles of rank f and g respectively, $f > g$ such that

1. the degeneracy locus of ϕ has codimension $f - g + 1$.
2. $F := H_*^0(Z, \mathcal{F})$ and $G := H_*^0(Z, \mathcal{G})$ are free R -modules.

Consider the exact sequence

$$0 \longrightarrow \mathcal{B}_\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathcal{M}_\phi \longrightarrow 0.$$

\mathcal{B}_ϕ is called a Buchsbaum-Rim sheaf. It is reflexive (as a second syzygy module) of rank $r = f - g$. Let \mathcal{P} be a decomposable vector bundle of rank q , $1 \leq q < r$, and $\psi : \mathcal{P} \rightarrow \mathcal{B}_\phi$ be a “generalized section”. It induces a map $\wedge^q \psi^* : \wedge^q \mathcal{B}_\phi^* \rightarrow \wedge^q \mathcal{P}^*$ where $\wedge^q \mathcal{P}^*$ is a line bundle. Let $\delta_q := \wedge^q \psi^*$. Then we get a degeneracy locus S

$$\wedge^q \mathcal{B}_\phi^* \otimes (\wedge^q \mathcal{P}^*)^{-1} \xrightarrow{\delta_q \otimes id} \mathcal{O}_Z \rightarrow \mathcal{O}_S \rightarrow 0.$$

We furthermore assume that the generalized section ψ is regular, i.e. S has the expected codimension $r - q + 1$. Let $I(\psi)$ denote the saturated ideal of S , X the top-dimensional part of S , $J = J(\psi)$ its saturated ideal and $P := H_*^0(\mathbb{P}^n, \mathcal{P})$.

The aim of the story : Study the geometrical properties of S and X .

We cite parts of the main result obtained in [6].

Theorem 2 *Using the notation from above, one has the following:*

- If $r + q$ is odd then $X = S$ is arithmetically Cohen-Macaulay iff $q = 1$. In this case, X has Cohen-Macaulay type $\leq 1 + \binom{\frac{r}{2} + q - 1}{q - 1}$.
- If $r + q$ is even then X is arithmetically Cohen-Macaulay iff $q = 1, 2$. In case $q = 1$, X is arithmetically Gorenstein. If moreover $r < n$ then components of S has either codimension $r - q + 1$ or codimension $r + 1$.
- Moreover, there is a resolution

$$0 \rightarrow A_r \oplus C_r \rightarrow \dots \rightarrow A_1 \oplus C_1 \rightarrow I_X \otimes \wedge^q \mathcal{P}^* \rightarrow 0,$$

where

$$\begin{aligned} A_k &= \bigoplus_{\substack{i+2j=k+q-1 \\ q \leq i+j \leq \frac{r+q-1}{2}}} \wedge^i F^* \otimes S^j(G)^* \otimes S^{i+j-q}(P), \\ C_k &= \bigoplus_{\substack{i+2j=r+1-q-k \\ i+j \leq \frac{r-q}{2}}} \wedge^i F \otimes S^j(G) \otimes S^{r-q-i-j}(P) \otimes \wedge^f F^* \otimes \wedge^g G. \end{aligned}$$

Important techniques used in the proof: Local cohomology; the Eagon-Northcott complex, its dual complex and their homology; the Buchsbaum-Rim complex.

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Remark 3: In order to clear the fog in the jungle of wedges and symmetric powers in Theorem 2, we explicitly write down the resolution of I_X for the case of a regular section in a rk-5 Buchsbaum-Rim sheaf \mathcal{B}_ϕ on \mathbb{P}^6 (i.e. $q = 1, r = 5$, $R = k[z_0, \dots, z_6]$ and $c_1 := c_1(\mathcal{B}_\phi)$):

$$\begin{aligned} 0 \rightarrow R(-c_1) \rightarrow S^2(G)^* \oplus F(-c_1) \rightarrow F^* \otimes G^* \oplus G(-c_1) \oplus (\wedge^2 F)(-c_1) \rightarrow \\ \rightarrow (\wedge^2 F)^* \oplus G^* \oplus F \otimes G(-c_1) \rightarrow F^* \oplus S^2(G)(-c_1) \rightarrow I_X \rightarrow 0 \end{aligned}$$

Corollary 2 *In the assumptions of the previous theorem, let $R = k[z_0, z_1, \dots, z_n]$, $q = 1$ and r be an odd integer. Choose a homogeneous $t \times (t+r)$ -matrix A over R , defining a morphism between free modules $F \xrightarrow{A} G$. Suppose that the ideal of all $t \times t$ -minors of A has the expected codimension $r+1$. After sheafifying, we get an exact sequence*

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{B}_A \longrightarrow & \mathcal{F} & \xrightarrow{A} & \mathcal{G} & \longrightarrow \text{coker}(A) \longrightarrow 0. \\ & \parallel & & \parallel & \\ & \bigoplus_{i=1}^{t+r} \mathcal{O}_{\mathbb{P}^n}(a_i) & \xrightarrow{A} & \bigoplus_{j=1}^t \mathcal{O}_{\mathbb{P}^n}(b_j) & \end{array}$$

Let s be a regular section of \mathcal{B}_A , $S := Z(s)$ and X be the top-dimensional part of S . Then X is arithmetically Gorenstein of codimension r .

Using the Buchsbaum-Rim resolution of \mathcal{B}_A , one can compute $\deg(X) = c_r(\mathcal{B}_A)$ in terms of the twists a_i and b_j of \mathcal{F} and \mathcal{G} respectively in a similar fashion as for the Gorenstein points in \mathbb{P}^3 .

Remark 4: It can be shown that the Buchsbaum-Rim sheaves on \mathbb{P}^n are exactly the reflexive Eilenberg-MacLane sheaves \mathcal{B} of rank $r \leq n$ such that $H_*^{n-r+1}(\mathbb{P}^n, \mathcal{B})^\vee$ is a Cohen-Macaulay R -module of dimension $\leq n-r$.

This implies immediately:

Corollary 3 *Let \mathcal{E} be a (non-split) vector bundle of odd rank $3 \leq r \leq n$ on \mathbb{P}^n with vanishing intermediate cohomology, with the exception of $H_*^{n-r+1}(\mathbb{P}^n, \mathcal{E}) \neq 0$. Let s be a regular section of \mathcal{E} . Then the top-dimensional component of the r -codimensional zero-locus $Z(s) \subset \mathbb{P}^n$ is arithmetically Gorenstein.*

4. Relation to Good Determinantal Subschemes

As an additional motivation for the study of Buchsbaum-Rim sheaves \mathcal{B}_ϕ , we want to mention that the loci of regular sections of their duals \mathcal{B}_ϕ^* actually correspond to a certain class of determinantal subschemes of projective space.

Definition 1 *A subscheme $V = V_t(A) \subset \mathbb{P}^n$ is called determinantal if it is given as the zero set of all $t \times t$ -minors of some homogeneous $g \times f$ -matrix A over $R = k[z_0, \dots, z_n]$. If V_t has the expected codimension $(g-t+1)(f-t+1)$ it is called a standard determinantal subscheme.*

In codimension 2 the situation is simple.

Theorem 3 (Hilbert-Burch) *A codimension 2 subscheme in \mathbb{P}^n is standard determinantal if and only if it is arithmetically Cohen-Macaulay.*

Now let A be a $t \times (t+r)$ -matrix such that $V_t(A)$ has the expected codimension $r+1$. Suppose that one can delete a generalized row from A such that the ideal of maximal minors of the resulting $(t-1) \times (t+r)$ -matrix has the expected codimension $r+2$. Then V_t is called a *good determinantal* subscheme.

Every complete intersection in \mathbb{P}^n is a good determinantal subscheme. In particular, all codimension 2 Gorenstein schemes are good determinantal subschemes.

Examples

The rational normal curve $V_2(A) \subset \mathbb{P}^n$ given by

$$A = \begin{pmatrix} z_0 & z_1 & z_2 & \cdots & z_{n-1} \\ z_1 & z_2 & z_3 & \cdots & z_n \end{pmatrix}$$

is a determinantal subscheme. Let

$$B = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}.$$

Then $V_2(B) \subset \mathbb{P}^3$ is standard determinantal but not good determinantal.

The main result in the context with Buchsbaum-Rim sheaves is:

Theorem 4 ([4]) *Let X be a subscheme of \mathbb{P}^n with $\text{codim}(X) \geq 2$. The following statements are equivalent :*

- *X is a good determinantal subscheme of codimension $r + 1$.*
- *X is a zero locus $Z(s)$ of a regular section $s \in H^0(\mathbb{P}^n, \mathcal{B}_\phi^*)$ of the dual \mathcal{B}_ϕ^* of a Buchsbaum-Rim sheaf of rank $r + 1$.*
- *X is standard determinantal and locally a complete intersection outside of some subscheme $Y \subset X$ of codimension $r + 2$ in \mathbb{P}^n .*

Corollary 4 *A zero-scheme in \mathbb{P}^3 is good determinantal if and only if it is standard determinantal and a local complete intersection.*

Let us mention that in the example above the ideal of $V_2(B)$ is $(z_1, z_2, z_3)^2$. It is easy to see that it is not a local complete intersection.

5. Implementation in Singular and Examples

Now we want to describe how to put the construction method into practice. The second author of this article implemented it in the computer algebra system *Singular*.¹ All the procedures in this and the following section will soon be available as the Singular library `buchsrim.lib`. Chris Peterson wrote similar procedures for Macaulay.²

The first question is: How can we compute a global section of a Buchsbaum-Rim sheaf \mathcal{B}_A of rank r in an algorithmical way?

We start with a homogeneous $t \times (t+r)$ matrix A . Recall that A is a block matrix

¹Singular is available at <http://www.singular.uni-kl.de/>

²Macaulay 2 can be downloaded from <http://www.math.uiuc.edu/Macaulay2/>

with blocks A_{ij} consisting of homogeneous polynomials in $R = k[z_0, \dots, z_n]$. It can be considered as a part of the exact sequence

$$0 \longrightarrow \mathcal{B}_A \longrightarrow \mathcal{F} \xrightarrow{A} \mathcal{G}.$$

Twist this sequence until we get global sections of \mathcal{F} . Now apply $H_*^0(\bullet)$. We get

$$0 \longrightarrow H_*^0(\mathcal{B}_A) \longrightarrow F \xrightarrow{A} G.$$

Let $H \xrightarrow{B} F \xrightarrow{A} G$ be a syzygy sequence of A . The space of global sections of \mathcal{B}_A is just the degree zero component of $Im(B)$. So, we have to compute a matrix representation B of the first syzygy module of A , say via the command `syz(A)`. Next, we take a linear combination $s = \sum_i f_i B_i$ of the columns B_i , where the coefficients f_i are *randomly chosen* homogeneous forms of some high degree d . The ideal of the vanishing locus $Z(s)$ of the section is of course the transpose of the column vector s .

Given a homogeneous matrix A and the degree d of the forms f_i , the Singular procedure `section(module A, int d)` generates such a random section s and returns the ideal of $Z(s)$. One should check whether the section s is regular, e.g. by testing `dim(std(section(A,d))) = n - r + 1`.

The most expensive part is the isolation of the top-dimensional components X of $Z(s)$. There are several ways to perform this computation:

$$I_X = Ann_R Ext_R^r(R/I_{Z(s)}, R)$$

This can be computed using the standard scripts for *Ext* and *Ann* in Singular, but especially for high codimension r , this method is quite time-consuming: One has to compute about r Gröbner basis in order to get I_X .

Instead, we use the following trick from liaison theory: Choose a regular sequence J in $I_{Z(s)}$ of length r (a randomly chosen sequence of homogeneous elements of high degree is generically regular). Then the double ideal quotient $(J : (J : I_{Z(s)}))$ is the saturated ideal I_X (Singular procedure: `top(ideal i)`). This method is much faster, but one should definitely try to find another, more effective way to compute I_X .

The procedure `br(...)` includes everything described above and returns the ideal of the top-dimensional part of a regular section of the Buchsbaum-Rim sheaf \mathcal{B}_A associated to a homogeneous matrix A . This matrix can either be chosen randomly or be a specific one. As an option, one can obtain a detailed protocol of all the computations.

Example 2: We give an annotated and slightly edited *Singular* session which produces an arithmetically Gorenstein curve of degree 21 in \mathbb{P}^6 .

Consider a 1×6 -matrix A of randomly chosen linear forms on \mathbb{P}^6 such that

$$0 \longrightarrow \mathcal{B}_A \longrightarrow 6 \mathcal{O}_{\mathbb{P}^6}(2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^6}(3) \longrightarrow \mathcal{M}_A \longrightarrow 0 \quad (1)$$

is exact with $\text{Supp}(\mathcal{M}_A) = Z$, $\dim Z = 0$. The first Chern class of the reflexive rk-5 sheaf \mathcal{B}_A equals 9. Using the Buchsbaum-Rim resolution of \mathcal{B}_A , we obtain $\deg S = \deg X = 21$.

```
>ring r=32003,z(0..6),dp;
>LIB "buchsrim.lib";      (load the library buchsrim.lib)
>ideal i=br(1,5,1,2);

// We start with a random 1 x 6 matrix A of degree 1 forms:
// A[1,1],A[1,2],A[1,3],A[1,4],A[1,5],A[1,6]

// Check the codimension of Supp(Coker(A))...
// The codimension is 6 as expected.

// The vanishing locus Z(j) of a randomly chosen section of the
// kernel sheaf is given by:

// j[1]=z(0)*z(3)-12625*z(1)*z(3)+68*z(2)*z(3)+11333*z(3)^2+...
// j[2]=z(2)^2+15226*z(1)*z(3)+8747*z(2)*z(3)-12086*z(3)^2+...
//
// j[6]=z(0)*z(1)-3269*z(1)*z(3)+9019*z(2)*z(3)+1243*z(3)^2-...
// Its top-dimensional part Z(i) is given by:

// i[1]=z(0)*z(3)-12625*z(1)*z(3)+68*z(2)*z(3)+11333*z(3)^2+...
// i[2]=z(2)^2+15226*z(1)*z(3)+8747*z(2)*z(3)-12086*z(3)^2+...
// i[3]=z(1)*z(2)-7394*z(1)*z(3)+2944*z(2)*z(3)-4713*z(3)^2-...
// i[4]=z(0)*z(2)+309*z(1)*z(3)+2913*z(2)*z(3)+2392*z(3)^2-...
// i[5]=z(1)^2-14366*z(1)*z(3)+11492*z(2)*z(3)+15380*z(3)^2+...
// i[6]=z(0)*z(1)-3269*z(1)*z(3)+9019*z(2)*z(3)+1243*z(3)^2-...
// i[7]=z(0)*z(4)^2-11212*z(1)*z(4)^2+9631*z(2)*z(4)^2-...

> hilb(std(i)); (compute the Hilbert functions of a standard basis of i)

// 1    t^0
// -6    t^2
// 21    t^4
// -21   t^5
// 6     t^7
// -1    t^9

// 1 t^0 ( The coefficients of the 2nd Hilbert function form the h-vector )
// 5 t^1
// 9 t^2
// 5 t^3
// 1 t^4

// codimension = 5 ( Singular computes  $\text{codim}_{\mathbb{A}^7} Z(I)$  and  $\dim_{\mathbb{A}^7} Z(I)$  )
// dimension = 2
// degree = 21      (... as expected)
```

From the h -vector $h_{I_X} = (1, 5, 9, 5, 1)$ we deduce the arithmetic genus $p_a(X) = 20$, the Castelnuovo-Mumford regularity $\text{reg}(\mathcal{I}_X) = 5$ and $\mathcal{O}_X \cong \omega_X(-2)$. Using the command `mres(i,0)`, we compute a *minimal* free resolution of the ideal I_X :

$$\begin{aligned} 0 \longrightarrow R(-9) \longrightarrow R(-6) \oplus 6R(-7) \longrightarrow 21R(-5) \oplus R(-6) \longrightarrow \\ \longrightarrow R(-3) \oplus 21R(-4) \longrightarrow 6R(-2) \oplus R(-3) \longrightarrow I_X \longrightarrow 0 \end{aligned}$$

Convince yourself that this is exactly the resolution from Remark 3 which is therefore minimal in this case.

Example 3: A Gorenstein threefold of degree 13 in \mathbb{P}^6 .

Choose the matrix $A = (a_{11}, a_{12}, a_{13}, a_{14})$ of quadratic forms in the exact sequence

$$0 \longrightarrow \mathcal{B}_A \longrightarrow 4\mathcal{O}_{\mathbb{P}^6}(3) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^6}(5) \longrightarrow \mathcal{M}_A \longrightarrow 0$$

randomly and in such a way that the vanishing locus has the expected codimension 4. Note that $c_1(\mathcal{B}_A) = 7$ and $\deg(X) = c_3(\mathcal{B}_A) = 13$. Again, we present an edited Singular session:

```
>ring r=23,z(0..6),dp;
>LIB "buchsrin.lib";
>ideal i=br(1,3,2,3);
```

We compute the two Hilbert functions of a Gröbner basis of the ideal $i = I_X$.

```
>hilb(std(i));

// 1 t^0
// -1 t^2
// -4 t^3
// 4 t^4
// 1 t^5
// -1 t^7

// 1 t^0
// 3 t^1
// 5 t^2
// 3 t^3
// 1 t^4

// codimension = 3
// dimension = 4 (affine...)
// degree = 13
```

As expected, the arithmetically Gorenstein threefold X has degree 13. Its h -vector is $h_{I_X} = (1, 3, 5, 3, 1)$. Furthermore, we get $\mathcal{O}_X \cong \omega_X$ and $\text{reg}(\mathcal{I}_X) = 5$. A minimal free resolution of I_X obtained via `mres(i,0)` is

$$0 \longrightarrow R(-7) \longrightarrow 4R(-4) \oplus R(-5) \longrightarrow R(-2) \oplus 4R(-3) \longrightarrow I_X \longrightarrow 0,$$

as stated in Theorem 2.

6. Application to Gorenstein Liaison

We shall apply the construction method also to deal with the following problem: Let V be equidimensional scheme in \mathbb{P}^n of odd codimension c . We want to find an arithmetically Gorenstein subscheme X of the same codimension c , which contains V . Thus, we get a direct G -link $V \overset{X}{\sim} W$, where W denotes the residue, i.e. the scheme associated to the saturated ideal $(I_X : I_V)$. Surely, we can easily find a complete intersection with the desired property. But from the point of view of Gorenstein liaison we are interested in arithmetically Gorenstein schemes which are not complete intersections.

Algorithm

1. Choose a rk- c Buchsbaum-Rim sheaf \mathcal{B}_ϕ on \mathbb{P}^n .

$$0 \longrightarrow \mathcal{B}_\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathcal{M}_\phi \longrightarrow 0.$$

2. Choose a regular section s of $\mathcal{B}_\phi(j)$ (for some shift $j \in \mathbb{Z}$), which is also in $H_*^0(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{I}_V)$. Then the zero set $Z(s)$ of s contains V .
3. Compute the top-dimensional part X of the $Z(s)$. Then due to Corollary 2, X is arithmetically Gorenstein of codimension c **containing** V .

Remark 5:

We can always find a $j \in \mathbb{Z}$ such that $H^0(\mathbb{P}^n, \mathcal{B}_\phi(j)) \cap H^0(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{I}_V) \neq 0$.

Let us consider an example.

Example 4: Let F be the Veronese surface in \mathbb{P}^5 given by the 2×2 -minors of the matrix

$$\begin{pmatrix} z_0 & z_3 & z_4 \\ z_3 & z_1 & z_5 \\ z_4 & z_5 & z_2 \end{pmatrix}$$

Therefore F is arithmetically Cohen-Macaulay (cf. [1], p. 84). Consider the exact sequence

$$0 \longrightarrow \mathcal{B}_\phi \longrightarrow 4 \mathcal{O}_{\mathbb{P}^5}(2) \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^5}(3) \longrightarrow \mathcal{M}_\phi \longrightarrow 0,$$

where $\phi := (z_1, z_2, z_3, z_4)$ and the cokernel \mathcal{M}_ϕ is supported on the line $L = Z(z_1, z_2, z_3, z_4) \subset \mathbb{P}^5$. The first syzygy module of ϕ is

$$B = \begin{pmatrix} 0 & 0 & 0 & -z_2 & -z_3 & -z_4 \\ 0 & -z_3 & -z_4 & z_1 & 0 & 0 \\ -z_4 & z_2 & 0 & 0 & z_1 & 0 \\ z_3 & 0 & z_2 & 0 & 0 & z_1 \end{pmatrix}.$$

View the saturated ideal of the surface F

$$I_F = (z_3 z_4 - z_0 z_5, z_1 z_4 - z_3 z_5, z_2 z_3 - z_4 z_5, z_1 z_2 - z_5^2, z_0 z_2 - z_4^2, z_0 z_1 - z_3^2)$$

as a 1×6 -matrix. The global sections of $4 \mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_F$ are linear combinations of columns of the matrix $I_F \otimes \text{Id}_4$. To find a common section s of $\mathcal{B}_\phi(d)$ and $4 \mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_F$, we consider the intersection of the modules B and $I_F \otimes \text{Id}_4$. Then s is just a linear combination of the columns of the matrix $B \cap (I_F \otimes \text{Id}_4)$ where the coefficients are homogeneous d -forms in z_0, \dots, z_5 whose degree d has to be chosen large enough.

Computing this with Singular, one observes that the vanishing locus $Z(s)$ of a random section s is for example given by the ideal:

$$\begin{aligned} I = & (z_1 z_4 - z_3 z_5, z_2 z_3 - z_4 z_5, z_0 z_2 - z_4^2, z_0 z_1 - z_3^2, \\ & z_3 z_4^2 - z_0 z_4 z_5, z_3^2 z_4 - z_0 z_3 z_5) = I_F \cap (z_1, z_2, z_3, z_4) \cap (z_0, z_3, z_4). \end{aligned}$$

$Z(s)$ is consequently the union of F , containing the embedded line L , and the plane $H = Z(z_0, z_3, z_4)$. The regularity of the section s can be easily checked by computing the Hilbert function of a standard basis of I . We get $\dim Z(s) = 2$, $\deg Z(s) = 5$ and $h_I = (1, 3, 2, -1)$ as h -vector.

After isolating the top-dimensional part $X \subset Z(s)$, we get a Gorenstein surface containing the Veronese surface F . The saturated ideal of X is

$$I_X = (z_3 z_4 - z_0 z_5, z_1 z_4 - z_3 z_5, z_2 z_3 - z_4 z_5, z_0 z_2 - z_4^2, z_0 z_1 - z_3^2) = I_F \cap I_H.$$

The symmetric h -vector $h_{I_X} = (1, 2, 1)$ and the minimal free resolution

$$0 \longrightarrow R(-5) \longrightarrow 5 R(-3) \longrightarrow 5 R(-2) \longrightarrow I_X \longrightarrow 0$$

confirm that X is Gorenstein. The Veronese surface F is therefore G -linked to the plane H .

7. Generalized Buchsbaum-Rim Sheaves

It is a natural question to ask what happens if the degeneracy locus of $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ does not have the expected codimension $f - g + 1$. Then the associated Buchsbaum-Rim complex is no longer acyclic and the sections of the kernel sheaves often lose the nice geometrical properties mentioned in Theorem 2.

We restrict our attention to a particular class of examples where the degeneracy locus is an *almost complete intersection*:

Let $G \subset \mathbb{P}^n$ be a codimension 3 arithmetically Gorenstein subscheme with $\mathcal{O}_G \cong \omega_G(l)$. Choose a complete intersection X of type (d_1, d_2, d_3) containing G . Let V denote the residue of G under the CI-link, i.e. the variety associated to the ideal $(I_X : I_G)$, and let $\alpha := d_1 + d_2 + d_3$. Using the exact sequence

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_G \rightarrow \omega_V(l) \rightarrow 0,$$

the two resolutions

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-\alpha) \longrightarrow \bigoplus_{i < j} \mathcal{O}_{\mathbb{P}^n}(-d_i - d_j) \longrightarrow \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^n}(-d_i) \longrightarrow \mathcal{I}_X \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(l - n - 1) \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{I}_G \longrightarrow 0$$

(\mathcal{E}_1 and \mathcal{E}_2 are decomposable bundles of the same rank m) and a mapping cone, we get

$$\boxed{0 \longrightarrow \mathcal{K}_\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathcal{O}_V(d) \longrightarrow 0,} \quad (2)$$

where

$$\left. \begin{aligned} \mathcal{F} &= \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^n}(d - d_i) \oplus \mathcal{O}_{\mathbb{P}^n}(n + 1 + d - l - \alpha) \\ \mathcal{G} &= \mathcal{O}_{\mathbb{P}^n}(d) \end{aligned} \right\} d > \max_i \{d_i\}.$$

The degeneracy locus of those morphisms ϕ has therefore codimension 3 and not 4 as expected. Note that the bundles \mathcal{F} have global sections.

Definition 2 We call such kernel sheaves \mathcal{K}_ϕ *generalized Buchsbaum-Rim sheaves*.

They have a free resolution

$$0 \longrightarrow \mathcal{E}_1^*(d - \alpha) \longrightarrow \bigoplus_{i < j} \mathcal{O}_{\mathbb{P}^n}(d - d_i - d_j) \oplus \mathcal{E}_2^*(d - \alpha) \longrightarrow \mathcal{K}_\phi \longrightarrow 0 \quad (3)$$

and are reflexive rk-3 sheaves (as 2nd syzygy sheaves).

Our interest now focuses on properties of regular sections s of generalized Buchsbaum-Rim sheaves \mathcal{K}_ϕ . Unfortunately, their zero-loci $Z(s)$ are no longer arithmetically Gorenstein. However, one can still determine a resolution of $I_{Z(s)}$.

Theorem 5 *Let s be a regular section of a generalized Buchsbaum-Rim sheaf \mathcal{K}_ϕ on \mathbb{P}^3 . Then the zero locus $Z(s)$ is an almost complete intersection and its saturated ideal $I_{Z(s)} \subset R = k[z_0, z_1, z_2, z_3]$ has a free resolution*

$$\begin{aligned} 0 \longrightarrow R(\alpha - d - b) \oplus E_1^*(-b) &\longrightarrow \bigoplus_{i=1}^3 R(d_i - b) \oplus R(-d) \oplus E_2^*(-b) \longrightarrow \\ &\longrightarrow \bigoplus_{i=1}^3 R(d_i - d) \oplus R(d - b) \longrightarrow I_{Z(s)} \longrightarrow 0, \end{aligned}$$

where $E_i := H_*^0(\mathbb{P}^3, \mathcal{E}_i)$ and $b := 2d - \alpha - l + 4$.

Proof: Let \mathcal{A} denote the image of ϕ in (2). Splitting the sequence into two short ones and applying $\mathcal{H}om(\bullet, \mathcal{O}_{\mathbb{P}^3})$, we get

$$0 \longrightarrow \mathcal{H}om(\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-d)) \longrightarrow \mathcal{G}^* \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{E}xt^1(\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-d)) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{K}_\phi^* \longrightarrow \mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow 0.$$

$\mathcal{E}xt_{\mathcal{O}_{\mathbb{P}^3}}^i(\mathcal{O}_V, \omega_{\mathbb{P}^3})$ vanishes for $0 \leq i < 3$ and therefore $\mathcal{G}^* \cong \mathcal{A}^*$ implies $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}(d)$. Hence the short exact sequence

$$0 \longrightarrow \mathcal{K}_\phi \longrightarrow \mathcal{F} \longrightarrow \mathcal{A} \longrightarrow 0$$

shows that \mathcal{K}_ϕ is even locally free. Furthermore, we obtain a free resolution

$$0 \longrightarrow G^* \longrightarrow F^* \longrightarrow H_*^0(\mathbb{P}^3, \mathcal{K}_\phi^*) \longrightarrow 0$$

because $\mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_{\mathbb{P}^3}) = 0$ and $H_*^1 \mathcal{G} = 0$. Let

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{K}_\phi \longrightarrow \mathcal{C} \longrightarrow 0 \quad (4)$$

be the exact sequence induced by the regular section s . Dualizing it and using $\mathcal{E}xt^1(\mathcal{K}_\phi, \mathcal{O}_{\mathbb{P}^3}) = 0$, we get

$$0 \longrightarrow \mathcal{C}^* \longrightarrow \mathcal{K}_\phi^* \longrightarrow \mathcal{I}_{Z(s)} \longrightarrow 0.$$

Dualizing a second time, we realize that \mathcal{C} is a reflexive rank 2 sheaf. Therefore, $\mathcal{C}^* \cong \mathcal{C}(-c_1)$. Now use (3) and (4) in order to check that $H_*^1(\mathbb{P}^3, \mathcal{K}_\phi) \cong H_*^1(\mathbb{P}^3, \mathcal{C}) = 0$ and to get a free resolution of $H_*^0 \mathcal{C}(-c_1)$ via a mapping cone:

$$\begin{aligned} 0 \rightarrow R(-c_1) \oplus E_1^*(d - \alpha - c_1) &\rightarrow \bigoplus_{i=1}^3 R(d + d_i - \alpha - c_1) \oplus E_2^*(d - \alpha - c_1) \\ &\rightarrow H_*^0 \mathcal{C}(-c_1) \rightarrow 0 \end{aligned}$$

Consequently, we obtain the following diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \begin{array}{c} R(-c_1) \\ \oplus \\ E_1^*(d - \alpha - c_1) \end{array} & \longrightarrow & \begin{array}{c} \bigoplus_{i=1}^3 R(d + d_i - \alpha - c_1) \\ \oplus \\ E_2^*(d - \alpha - c_1) \end{array} & \longrightarrow & H_*^0 \mathcal{C}(-c_1) \longrightarrow 0 \\ & & & & \downarrow & & \\ 0 & \longrightarrow & G^* & \longrightarrow & F^* & \longrightarrow & H_*^0(\mathcal{K}_\phi^*) \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & \mathcal{I}_{Z(s)} \\ & & & & & & \downarrow \\ & & & & & & 0. \end{array}$$

Another application of the mapping cone lemma and the fact that $c_1(\mathcal{C}) = c_1(\mathcal{K}_\phi) = \deg(\mathcal{F}) - \deg(\mathcal{G})$ is equal to $3d - 2\alpha - l + 4 = d - \alpha + b$ imply the claim. Note that F is a free R -module of rank 4. Thus, $Z(s)$ is an almost complete intersection of type $(d - d_1, d - d_2, d - d_3, d - \alpha - l + 4)$.

◇

Example 5: As Gorenstein scheme G , let us take 5 points in \mathbb{P}^3 in general position. Their saturated ideal I_G can be obtained using a regular section s of the Buchsbaum-Rim sheaf

$$0 \longrightarrow B_\phi \longrightarrow 4 \mathcal{O}_{\mathbb{P}^3}(2) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^3}(3) \longrightarrow 0,$$

where $\psi = (z_0, z_1, z_2, z_3)$. Note that $\mathcal{B}_\psi = \Omega_{\mathbb{P}^3}^1(3)$ and $c_3(\mathcal{B}_\psi) = 5$. According to Theorem 1, there exists a free resolution

$$0 \longrightarrow R(-5) \longrightarrow 5 R(-3) \longrightarrow 5 R(-2) \longrightarrow I_G \longrightarrow 0.$$

Now we choose a complete intersection X of three cubic forms containing G . The residue V are 22 points in \mathbb{P}^3 . Their saturated ideal I_V is an almost complete intersection and has the following minimal free resolution:

$$0 \longrightarrow 5 R(-7) \longrightarrow 8 R(-6) \longrightarrow 3 R(-3) \oplus R(-4) \xrightarrow{\phi} I_V \longrightarrow 0.$$

Compare it with (2) and (3). Thus, the “data” for this example is

$$\mathcal{E}_1 = 5 \mathcal{O}_{\mathbb{P}^3}(-2), \mathcal{E}_2 = 5 \mathcal{O}_{\mathbb{P}^3}(-3), d_1 = d_2 = d_3 = 3, \alpha = 9 \text{ and } l = -1.$$

We choose $d = 6$. Using the Singular command `mres(section(syz(ϕ), 3), 0)` we get the a minimal free resolution of the degree 13 zero-locus $Z(s)$ of a regular section $s \in H^0(\mathbb{P}^3, \mathcal{K}_\phi)$:

$$0 \longrightarrow 4 R(-6) \longrightarrow 7 R(-5) \longrightarrow 3 R(-3) \oplus R(-2) \longrightarrow I_{Z(s)} \longrightarrow 0$$

We recover exactly this sequence after deleting the “ghost-summand” $R(-5) \oplus R(-6)$ in the (non-minimal) free resolution in Theorem 5.

Remark 6: The authors believe that it is straightforward to show that an analogon of Theorem 5 holds for regular sections s of generalized Buchsbaum-Rim sheaves \mathcal{K}_ϕ on \mathbb{P}^n , $n \geq 4$. One should use the fact that the Cohen-Macaulay type and the resolution are preserved under general hyperplane sections.

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